Introduction to inter-temporal (dynamic) optimization
(Handout of lecture #3, Ekonomi SDA dan Lingkungan, Arief Anshory Y)
What is optimization?
Maximization or Minimization of certain objectives (function)
e.g.
Maximizing utility or choosing number of goods consumed that maximize utility
Choosing number of output produced that maximizing profit or minimizing cost
Static optimization without constraint
One variable input (Labor, L) production function:

$$Q(L) := a + b \cdot L + c \cdot L^2 + d \cdot L^3$$

$a \equiv 10$

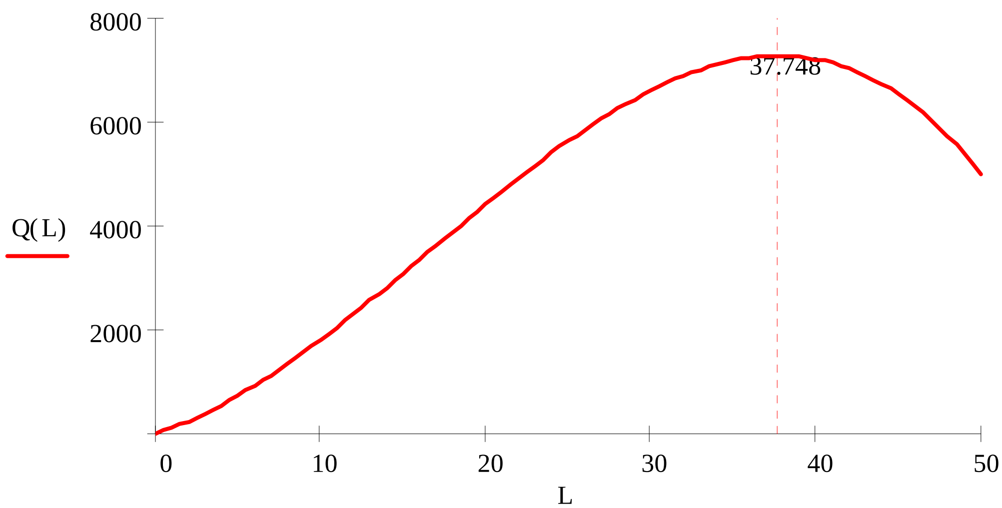
$b \equiv 100$

$c \equiv 10$

$d \equiv -0.20$

$L \equiv 0, 0.5 .. 50$

Choose L that maximize Q!



Find First Order Condition (FOC)

$$\frac{d}{dL} Q(L) = 0$$

$$\frac{d}{dL} Q(L) \rightarrow 100 + 20 \cdot L - .6 \cdot L^2$$

$$100 + 20 \cdot L - .6 \cdot L^2 = 0 \text{ solve}, L \rightarrow \begin{bmatrix} -4.4151844011225288 \\ 37.748517734455862214 \end{bmatrix}$$

INTUITION OF THE FOC:
The conditions that must be hold for the objective function to be maximized, otherwise it won't be maximized (or it is not optimal).
Output will be maximized if Marginal Product of Labor (MPL) equal to zero meaning that is when L = L* = 37.75, using L < L* or L > L* output will be reduced.
Important assumption: firm maximizing output (objective function), the shape of the

production function (diminishing MPL).

Static optimization with constraint (method of Lagrange Multiplier)

Maximize utility

$$U(X1, X2)$$

subject to

$$P1 \cdot X1 + P2 \cdot X2 = Y$$

$$Lg = U(X1, X2) - \lambda \cdot (P1 \cdot X1 + P2 \cdot X2 - Y)$$

Max Lg

First Order Condition

$$\frac{d}{d X1} U - \lambda \cdot P1 = 0$$

$$\frac{d}{d X2} U - \lambda \cdot P2 = 0$$

$$P1 \cdot X1 + P2 \cdot X2 - Y = 0$$

It follows that

$$\frac{\frac{d}{d X1} U}{P1} = \frac{\frac{d}{d X2} U}{P2}$$

or

$$\frac{MU_{X1}}{MU_{X2}} = \frac{P1}{P2}$$

Always understand the intuition behind this FOC!

Example, do you comprehend this basic FOC from microeconomics principle?

$$\frac{MU_{X1}}{P1} = \frac{MU_{X2}}{P2}$$

example (to be added)

Dynamic Optimization

Simplest case, two period: period 0 (now) and period 1 (later)

Suppose

$$Q$$

is extraction which is costless (for simplicity)

We face (inverse) demand function P(Q)

We want to maximize profit/revenue

$$R(Q) = P(Q) \cdot Q$$

If there is onle one period (now) then we choose Q that follow this condition:

$$\frac{d}{d Q} R(Q) = 0$$

or

$$\left(\frac{d}{dQ}P(Q)\right)\cdot Q + P(Q) = 0$$

or

Marginal Revenue = MR =0

If there are two period, and we have to extract/produce in both period how much is extraction for each period Q0 and Q1?

Objective function become

$$V(Q_0, Q_1) = R(Q_0) + R(Q_1)$$

Standard FOC applied (without constraint)

Lets use natural resource example!

We have stock of reserves which is fixed of S, we could extract it without cost (for simplicity), we follow standard demand function P(Q).

Static (single period): how much we extract?

there is a constraint

$$Q \le S$$

We could not extract more than S, but we could left some reserves in the ground.

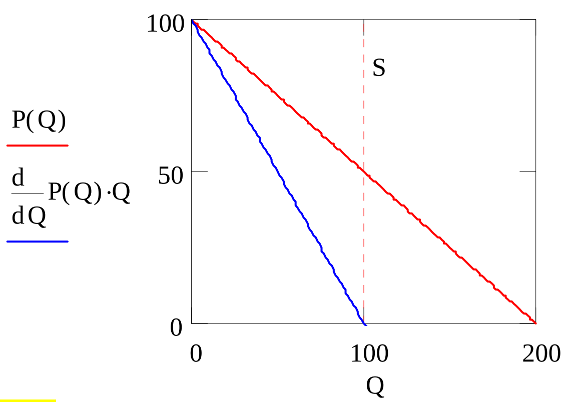
e.g.

$$P(Q) := \alpha - \beta \cdot Q$$

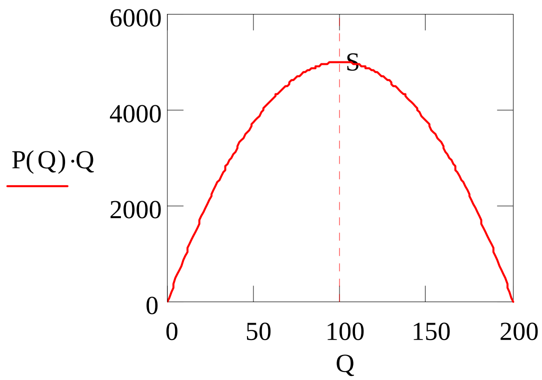
$$\alpha \equiv 100$$

$$\beta \equiv 0.5$$

$$Q := 0 .. 200$$



$$S \equiv 100$$



What if there are two periods?
Maximize

$$R_0(Q_0) + R_1(Q_1)$$

subject to

$$Q_0 + Q_1 = S$$

Reserves must not be left
in the ground
Max

$$R_0(Q_0) + R_1(Q_1) - \lambda \cdot (Q_0 + Q_1 - S)$$

FOC

$$\frac{d}{dQ_0} R_0 - \lambda = 0$$

$$\frac{d}{dQ_1} R_1 - \lambda = 0$$

$$Q_0 + Q_1 = S$$

It follows that

$$\frac{d}{dQ_0} R_0 = \frac{d}{dQ_1} R_1$$

or

$$MR_0 = MR_1$$

Intuition?
With the same revenue function R(Q)=P(Q)Q, it follows that

$$Q_0 = Q_1$$

Thus
S should be equally divided between two periods
This is the only condition that is optimal (optimization)
Introducing discount rate
Objective function: total revenue/profit
maximizing present value of revenue

$$R_0 + \frac{1}{1+r} \cdot R_1$$

subject to

$$R_0 + R_1 = S$$

Applied for both period:

$$R = P(Q) \cdot Q$$

$$P(Q) := \alpha - \beta \cdot Q$$

our problem then ... choose

Q_0

and

Q_1

that

maximize

$$R(Q_0) + \frac{R(Q_1)}{1+r}$$

subject to

$$Q_0 + Q_1 = S$$

$$lg = R(Q_0) + \frac{R(Q_1)}{1+r} - \lambda \cdot (Q_0 + Q_1 - S)$$

FOC

$$\frac{d}{dQ_0} R(Q_0) - \lambda = 0$$

$$\left(\frac{d}{dQ_1} R(Q_1)\right) \cdot \left(\frac{1}{1+r}\right) - \lambda = 0$$

$$Q_0 + Q_1 = S$$

it follows that

$$MR_0 = \frac{MR_1}{1+r}$$

or

$$MR_1 = (1+r) \cdot MR_0$$

Hotelling Rule!

In a competitive world, where

$$R = P \cdot Q$$

that is

P

is independent of Q

$$MR = P$$

Hotelling rule become

$P_1 = P_0(1+r)$

at multiple discrete period, it become

$P_t = P_0(1+r)^t$

at multiple continuous period, it become

$P(t) = P_0 \cdot e^{r \cdot t}$

What is the intuition behind Hotelling Rule?

Note:Hotelling rule still applied when the objective function is maximizing social welfare that is consumer surplus minus social cost (see note, next lecture!)

Application of Hotelling Rule, see simulation!

General Problem of Dynamic Optimization (The principle only ...)

Discrete time

t = 0,1,2, ..., T

x_t

= state variable (we can not control directly, but we can indirectly, for example, stock of reserves)

y_t

= control or instrument variable (we can control, and in turn will affect state variable)

$V_t = V(x_t, y_t, t)$

Objective function, e.g. net economic return e.g. profit/welfare

$F(x_T)$

final function, what do we want at termintal time T

$x_{t+1} - x_t = f(x_t, y_t)$

difference equation defining how state variable change

The problem then (without final function, for simplicity)

Maximize

$V_0 + V_1 + V_2$

+ .. +

V_T

or

$$\sum_{t=0}^{T} V(x_t, y_t, t)$$

subject to

$x_{t+1} - x_t = f(x_t, y_t)$

$x_0 = a$

given

How much are

y_0

y_1

....

y_T

we can use lagrange multiplier ...

for example

t=0,1,2,3

maximize

$$V(.)_0 + V(.)_1 + V(.)_2 + V(.)_3$$

subject to

$$x_1 - x_0 = f(x_0, y_0)$$

$$x_2 - x_1 = f(.)$$

$$x_3 - x_2 = f(.)$$

$$lg = V(.)_0 + V(.)_1 + V(.)_2 + V(.)_3 + \lambda_1 \cdot (f(.) - x_1 + x_0) + \lambda_2 \cdot (f(.) - x_2 + x_1) + \lambda_3 \cdot (f(.) - x_3 + x_2)$$

or

$$lg = \sum_{t=0}^{3} V(.)_t + \sum_{t=1}^{3} \lambda_t \cdot (f(.) - x_t + x_{t-1})$$

FOC

differentiate

lg

with respect to

y_0, y_1, y_2, y_3

set equal to zero

differentiate

lg

with respect to

x_1, x_2, x_3

set equal to zero

differentiate

lg

with respect to

$\lambda_1, \lambda_2, \lambda_3$

set equal to zero

We have 10 unknown variable with 10 equation, we can in principle solve this!

In continuous form

maximize

$$\int_0^T V(x(t), y(t), t)\, dt$$

subject to

$$\frac{d}{dt}x(t) = f(x(t), y(t), t)$$

{y(t)}

There is a method called Hamiltonian (which is beyond our discussion)

Another example from natural resource problem .. Hotelling Rule again

We want to maximize total profit stream

$\Pi_0 + \Pi_1$

+ .... +

Π_T

or

$$\sum_{t=0}^{T} \Pi_t$$

where

$$\Pi_t = P_t \cdot Q_t - c(Q_t)$$

but subject to

$Q_0 + Q_1$

+ ... +

$Q_T = S$

or

$$\sum_{t=0}^{T} Q_t = S$$

with discounting, become maximizing present value of profit stream

that is maximize

$$\sum_{t=0}^{T} \Pi_t \left(\frac{1}{1+r}\right)^t$$

$$lg = \sum_{t=0}^{T} \left(P_t \cdot Q_t - c(Q_t)\right) \cdot \left(\frac{1}{1+r}\right)^t + \lambda \cdot \left(S - \sum_{t=0}^{T} Q_t\right)$$

FOC

$$\frac{d}{dQ_t} lg = 0$$

$$\left(P_t - \frac{d}{dQ_t} c(.)\right) \cdot \left(\frac{1}{1+r}\right)^t - \lambda = 0$$

for all t = 0 .. T
or

$$\left(P_t - MC_t\right)\cdot\left(\frac{1}{1+r}\right)^t - \lambda = 0$$

across adjacent period

$$\left(P_{t+1} - MC_{t+1}\right)\cdot\left(\frac{1}{1+r}\right)^{t+1} - \lambda = 0$$

thus

$$\left(P_t - MC_t\right)\cdot\left(\frac{1}{1+r}\right)^t = \left(P_{t+1} - MC_{t+1}\right)\cdot\left(\frac{1}{1+r}\right)^{t+1}$$

$$P_t - MC_t = \left(P_{t+1} - MC_{t+1}\right)\cdot\left(\frac{1}{1+r}\right)$$

Lets RENT = P - MC

$$RENT_{t+1} = RENT_t\cdot(1+r)$$

or

$$RENT_{t+1} = RENT_t + r\cdot RENT_t$$

$$\frac{RENT_{t+1} - RENT_t}{RENT_t} = r$$

HOTELLING RULE!
HOTELLING RULE
Rent increase at rate of discount
if MC = 0, price increase at rate of discount